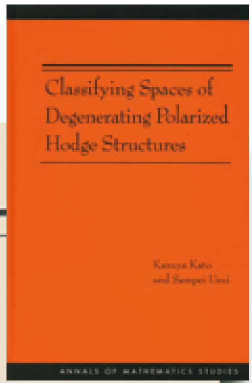
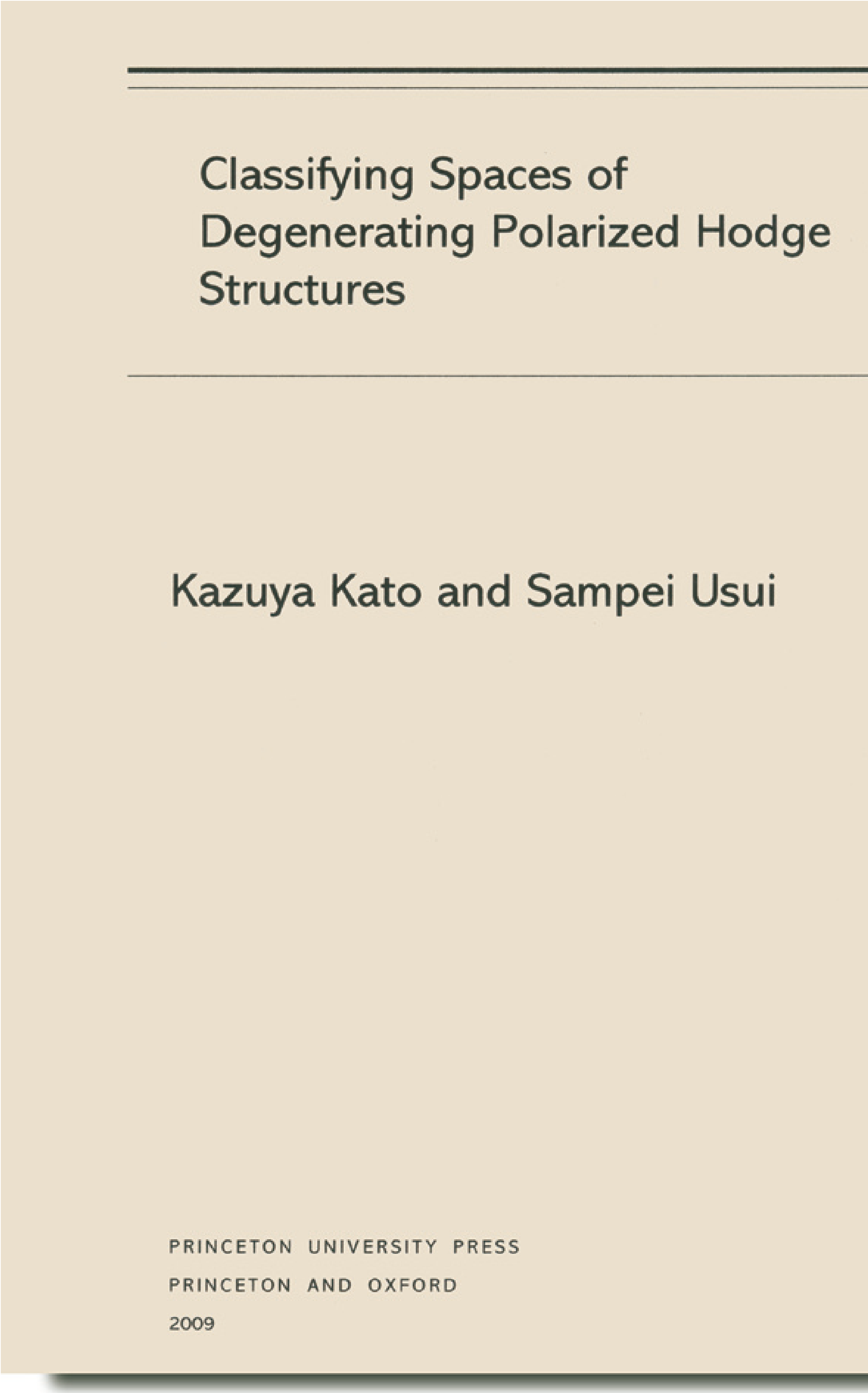
Classifying Spaces of Degenerating Polarized Hodge Structures

Paper in journals : this is the first page of a paper published in ***Annals of Mathematics Studies***.
[***Annals of Mathematics Studies***] **169**, 348 pages (2009)





Classifying Spaces of Degenerating Polarized Hodge Structures

Kazuya Kato and Sampei Usui

PRINCETON UNIVERSITY PRESS
PRINCETON AND OXFORD
2009

▲Reprinted with permission from Kato, K. and Usui, S., Classifying Spaces of Degenerating Polarized Hodge Structures, ***Annals of Mathematics Studies,*** 169 Copyright 2009, by Princeton University Press.

The following is a comment on the published paper shown on the preceding page.

On The Book "Classifying Spaces of Degenerating Polarized Hodge Structures" with Kazuya Kato

USUI Sampei
(Graduate School of Science)

Suugaku wa mugen enten kou kokoro
Koute kogarete harukana tabiji

by Kazuya Kato and Sampei Usui,
which was translated by Luc Illusie as

L'impossible voyage aux points à l'infini
N'a pas fait battre en vain le coeur du géomètre

Griffiths defined and studied in [G1] the classifying space D of polarized Hodge structures of fixed weight w and fixed Hodge numbers ($h^{p,q}$), and presented in [G2] a dream to add points at infinity to D. [KU1] is an announcement of our attempt to realize his dream, and this book is its full-detailed version.

In a special case where $w = 1$, $h^{1,0} = h^{0,1} = g$, and other $h^{p,q} = 0$, the classifying space D coincides with Siegel's upper half space $\mathfrak{h}_g$ of degree g. In this case, for a subgroup Γ of $\mathrm{Sp}(g,\mathbf{Z})$ of finite index, toroidal compactifications of $\Gamma\backslash\mathfrak{h}_g$ ([AMRT]) and the Satake-Baily-Borel compactification of $\Gamma\backslash\mathfrak{h}_g$ ([Sa], [BB]) are already constructed, where points at infinity often play more important roles than usual points. For example, in the simplest case $g = 1$, the Taylor expansion of a modular form at the standard cusp (i.e., the class of $\infty \in \mathbf{P}^1(\mathbf{Q})$ modulo Γ) of the compactified modular curve $\Gamma\backslash(\mathfrak{h}\cup\mathbf{P}^1(\mathbf{Q}))$ is called the q-expansion and is very important in the theory of modular forms.

The theory of these compactifications is included in a general theory of compactifications of quotients of symmetric Hermitian domains by the actions of discrete arithmetic groups. However, the classifying space D in general is rarely a symmetric Hermitian domain, and we can not use the general theory of symmetric Hermitian domains when we try to add points at infinity to D. In this book, we overcome this difficulty. We discuss two subjects.

Subject I. Toroidal partial compactifications and moduli of polarized logarithmic Hodge structures.

In this book, for general D, we construct a kind of toroidal partial compactification $\Gamma\backslash D_\Sigma$ of $\Gamma\backslash D$ associated to a fan Σ and a discrete subgroup Γ of $\mathrm{Aut}(D)$ satisfying a certain compatibility with Σ.

In the case $D = \mathfrak{h}_g$, the classes of polarized Hodge structures in $\Gamma\backslash\mathfrak{h}_g$ converge to a point at infinity of $\Gamma\backslash\mathfrak{h}_g$ when the polarized Hodge structures degenerate. As in [Sc], nilpotent orbits appear when polarized Hodge structures degenerate. In our definition of D_Σ for general D, a nilpotent orbit itself is viewed as a point at infinity. In order to do so, we use logarithmic structures introduced by Fontaine and Illusie and developed in [Kk1], [KkNc]. The theory of nilpotent orbits is regarded as a local aspect of the theory of polarized logarithmic Hodge structures (= PLH). A fundamental observation here is: (a nilpotent orbit) = (a PLH over a logarithmic point).

Our main theorem concerning Subject I is stated roughly as follows.

Theorem. $\Gamma\backslash D_\Sigma$ *is the fine moduli space of "polarized logarithmic Hodge structures" with a "*Γ*-level structure" whose "local monodromies are in the directions in* Σ*".*

In the classical case $D = \mathfrak{h}_g$, for a subgroup Γ of $\mathrm{Sp}(g,\mathbf{Z})$ of finite index and for a sufficiently big fan Σ, $\Gamma\backslash D_\Sigma$ is a toroidal compactification of $\Gamma\backslash\mathfrak{h}_g$. Already in this classical case, this theorem gives moduli-theoretic interpretations of the toroidal compactifications of $\Gamma\backslash\mathfrak{h}_g$.

For general D, the space $\Gamma\backslash D_\Sigma$ has a kind of complex structure, but a delicate point is that this space can have locally the shape of "complex analytic space with a slit" (for example, $\mathbf{C}^2$ minus $\{(0, z) \mid z \in \mathbf{C}, z \neq 0\}$), and hence it is often not locally compact. However it is very near to a complex analytic manifold, and we call it a "logarithmic manifold". Infinitesimal calculus is performed on $\Gamma\backslash D_\Sigma$ nicely. These phenomena are first examined in the easiest non-trivial case in [U1].

One motivation of Griffiths for adding points at infinity to D was the hope that the period map $\Delta^* \to \Gamma\backslash D$, associated to a variation of polarized Hodge structure on a punctured disc Δ^*, could be extended over the puncture. By using the above main theorem and the nilpotent orbit theorem of Schmid, we can actually extend the period map to $\Delta \to \Gamma\backslash D_\Sigma$ for some suitable fan Σ.

Subject II. The eight enlargements of D and the fundamental diagram.

In the classical case $D = \mathfrak{h}_g$, there is another compactification $\Gamma\backslash D_{\mathrm{BS}}$ of $\Gamma\backslash\mathfrak{h}_g$, for Γ a subgroup of $\mathrm{Sp}(g,\mathbf{Z})$ of finite index, called Borel-Serre compactification ([BS]). This is a real manifold with corners (like $\mathbf{R}^m_{\geq 0} \times \mathbf{R}^n$ locally).

For general D, by adding to D points at infinity of different kinds, we obtain eight enlargements of D with maps among them which form the following diagram.

Fundamental Diagram.

$$\begin{tikzcd}
 & & D_{\mathrm{SL}(2),\mathrm{val}} \arrow[r] \arrow[d] & D_{\mathrm{BS},\mathrm{val}} \arrow[d] \\
D_{\Sigma,\mathrm{val}} \arrow[d] & D^{\#}_{\Sigma,\mathrm{val}} \arrow[l] \arrow[r] \arrow[d] & D_{\mathrm{SL}(2)} & D_{\mathrm{BS}} \\
D_{\Sigma} & D^{\#}_{\Sigma} \arrow[l] & &
\end{tikzcd}$$

Note that the space D_Σ, which appeared in Subject I, sits at the left lower end of this diagram. Like nilpotent orbits, SL(2)-orbits also appear in the theory of degenerations of polarized Hodge structures ([Sc], [CKS]). This diagram tells how nilpotent orbits, SL(2)-orbits, and the theory of Borel-Serre are related. The left-hand side of the above diagram has Hodge-theoretic nature, and the right-hand side has the nature of theory of algebraic groups. These are related by the middle map $D^{\#}_{\Sigma,\mathrm{val}} \to D_{\mathrm{SL}(2)}$ which is a geometric interpretation of SL(2)-orbit theorem of Cattani-Kaplan-Schmid.

The eight spaces D_Σ, $D^{\#}_\Sigma$, $D_{\mathrm{SL}(2)}$, D_{BS}, $D_{\Sigma,\mathrm{val}}$, $D^{\#}_{\Sigma,\mathrm{val}}$, $D_{\mathrm{SL}(2),\mathrm{val}}$, $D_{\mathrm{BS},\mathrm{val}}$ in Fundamental Diagram are defined as the spaces of nilpotent orbits, nilpotent i-orbits, SL(2)-orbits, Borel-Serre orbits, valuative nilpotent orbits, valuative nilpotent i-orbits, valuative SL(2)-orbits, valuative Borel-Serre orbits, respectively. Roughly speaking, $\Gamma\backslash D_\Sigma$ is like an analytic manifold with slits, $D^{\#}_\Sigma$ and $D_{\mathrm{SL}(2)}$ are like real manifolds with corners and slits, D_{BS} is a real manifold with corners, $\Gamma\backslash D_{\Sigma,\mathrm{val}}$ and $D^{\#}_{\Sigma,\mathrm{val}}$ are the projective limits of "blowing-ups" of $\Gamma\backslash D_\Sigma$ and $D^{\#}_\Sigma$, respectively, associated to rational subdivisions of Σ, $D_{\mathrm{SL}(2),\mathrm{val}}$ and $D_{\mathrm{BS},\mathrm{val}}$ are the projective limits of certain "blowing-ups" of $D_{\mathrm{SL}(2)}$ and D_{BS}, respectively. The maps $\Gamma\backslash D^{\#}_\Sigma \to \Gamma\backslash D_\Sigma$ and $\Gamma\backslash D^{\#}_{\Sigma,\mathrm{val}} \to \Gamma\backslash D_{\Sigma,\mathrm{val}}$ are proper surjective maps, described by logarithmic structures, whose fibers are products of finite copies of $\mathbf{S}^1$.

In the classical case $D = \mathfrak{h}_g$, we have $D_{\mathrm{SL}(2)} = D_{\mathrm{BS}}$ and $D_{\mathrm{SL}(2),\mathrm{val}} = D_{\mathrm{BS},\mathrm{val}}$ in Fundamental Diagram. This is because, for such simple polarizes Hodge structures, all filters appeared in the associated set of monodoromy-weight filtrations are totally linearly ordered by inclusion and the Griffiths transversality becomes a vacant condition. Fundamental Diagram gives a relation between toroidal compactifications $\Gamma\backslash D_\Sigma$ of $\Gamma\backslash\mathfrak{h}_g$ and the Borel-Serre compactification $\Gamma\backslash D_{\mathrm{BS}}$ of $\Gamma\backslash\mathfrak{h}_g$. The Satake-Baily-Borel compactification sits under $\Gamma\backslash D_{\mathrm{SL}(2)} = \Gamma\backslash D_{\mathrm{BS}}$. Already in this classical case, these relations were not known before.

In this book, we study all these eight spaces. To prove the main theorem in Subject I and to prove that $\Gamma\backslash D_\Sigma$ has good properties such as Hausdorff property, nice infinitesimal calculus, etc., we need to consider all the eight spaces; we discuss from the right to the left in the Fundamental Diagram to deduce nice properties of $\Gamma\backslash D_\Sigma$, starting from nice properties of the Borel-Serre compactifications in [BS].

[U2], [U3] are some geometric applications of the present results. A new project [KNU], which is an evolution for logarithmic *mixed* Hodge structures, is in progress.

References

[AMRT] A. Ash, D. Mumford, M. Rapoport and Y. S. Tai, *Smooth compactification of locally symmetric varieties*, Math. Sci. Press, 1975.

[BB] L. Baily and A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math. **84** (1966).

[BS] A. Borel and J.-P. Serre, *Corners and arithmetic groups*, Comment. Math. Helv. **48** (1973).

[CKS] E. Cattani, A. Kaplan and W. Schmid, *Degeneration of Hodge structures*, Ann. of Math. **123** (1986).

[G1] P. A. Griffiths, *Periods of integrals on algebraic manifolds. I. Construction and properties of modular varieties*, Amer. J. Math. **90** (1968).

[G2] __________, *Periods of integrals on algebraic manifolds: Summary of main results and discussions of open problems*, Bull. Amer. Math. Soc. **76** (1970).

[Kk1] K. Kato, *Logarithmic structures of Fontaine-Illusie*, in "Algebraic analysis, geometry, and number theory", Johns Hopkins University Press, 1989.

[KkNc] K. Kato and C. Nakayama, *Log Betti cohomology, log étale cohomology, and log de Rham cohomology of log schemes over* **C**, Kodai Math. J. **22** (1999).

[KNU] K. Kato, C. Nakayama and S. Usui, *Classifying spaces of degenerating mixed Hodge structures*,, Part I (appeared), Part II (in preparation), Part III (in preparation), ...

[KU1] K. Kato and S. Usui, *Logarithmic Hodge structures and classifying spaces*, CRM Proc. & Lect. Notes **24** (1999).

[KU2] __________, *Borel-Serre spaces and spaces of* SL(2)*-orbits*, Advanced Studies in Pure Math. **36** (2002).

[Sa] I. Satake, *On compactifications of the quotient spaces for arithmetically defined discontinuous groups*, Ann. of Math. **72** (1960).

[Sc] W. Schmid, *Variation of Hodge structure: the singularities of the period mapping*, Invent. Math. **22** (1973).

[U1] S. Usui, *Complex structures on partial compactifications of arithmetic quotients of classifying spaces of Hodge structures*, Tôhoku Math. J. **47** (1995).

[U2] __________, *Images of extended period maps*, J. Alg. Geom. **15-4** (2006).

[U3] __________, *Generic Torelli theorem for mirror-quntic family*, Proc. Japan Acad., Ser. A, 84A (8) (2008).